Date — 30/6/18

Diff. Calculus

i) Sitansu Bondhopadyay (application of calculus) for part-I

ii) Diff. Calculus, Maity and Others

1st chapter :— ~~Higher~~ Successive Derivative (Higher order derivatives)

1) $Y = e^{ax}$

$y_1 = ae^{ax}$

$y_2 = a^2 e^{ax}$

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$y_n = a^n e^{ax}$

1) Let $y = e^{ax}$

Therefore, $y_1 = ae^{ax}$

$y_2 = a^2 e^{ax}$

$\therefore \boxed{y_n = a^n e^{ax}}$

2) Let $y = \sin(ax+b)$

$\therefore y_1 = a\cos(ax+b) = a\sin\left(1\cdot\frac{\pi}{2} + \overline{ax+b}\right)$

$y_2 = -a^2\sin(ax+b) = a^2\sin\left(2\times\frac{\pi}{2} + \overline{ax+b}\right)$

$y_3 = -a^3\cos(ax+b) = a^3\sin\left(3\times\frac{\pi}{2} + \overline{ax+b}\right)$

$y_4 = a^4\sin(ax+b) = a^4\sin\left(4\times\frac{\pi}{2} + \overline{ax+b}\right)$

$\therefore \boxed{y_n = a^n\sin\left(\frac{\pi}{2}n + \overline{ax+b}\right)}$

Exam:- $y = \sin x$

$\boxed{y_{100} = \sin x}$

Let $y = \sin x$

$\therefore y_{100} = \sin\left(100\times\frac{\pi}{2} + x\right) = \sin x$

3) Let $y = \cos(ax+b)$

$$y_1 = -a\sin(ax+b) = a\cos\left(\frac{\pi}{2} + \overline{ax+b}\right)$$

$$y_2 = -a^2\cos(ax+b) = a^2\cos\left(2\cdot\frac{\pi}{2} + \overline{ax+b}\right)$$

$$y_3 = a^3\sin(ax+b) = a^3\cos\left(3\cdot\frac{\pi}{2} + \overline{ax+b}\right)$$

$$\therefore \boxed{y_n = a^n\cos\left(n\frac{\pi}{2} + \overline{ax+b}\right)}$$

4) Let $y = \dfrac{1}{ax+b}$

$$\therefore y_1 = -\frac{a}{(ax+b)^2} = \frac{(-1)^1 \lfloor\underline{1}\, a}{(ax+b)^{1+1}}$$

$$y_2 = \frac{2a^2}{(ax+b)^3} = \frac{(-1)^2 \lfloor\underline{2}\, a^2}{(ax+b)^{2+1}}$$

$$y_3 = -\frac{2\times 3\, a^3}{(ax+b)^4} = \frac{(-1)^3 \lfloor\underline{3}\, a^3}{(ax+b)^{3+1}}$$

$$y_4 = \frac{2\times 3\times 4\times a^4}{(ax+b)^5} = \frac{(-1)^4 \lfloor\underline{4}\, a^4}{(ax+b)^{4+1}}$$

$$\therefore \boxed{y_n = \frac{(-1)^n \lfloor\underline{n}\, a^n}{(ax+b)^{n+1}}}$$

5) Let $y = \log(ax+b)$

$$\therefore y_1 = \frac{a}{ax+b} = \frac{(-1)^{1-1} \lfloor\underline{1-1}\, a^1}{(ax+b)^1}$$

$$y_2 = -\frac{a^2}{(ax+b)^2} = \frac{(-1)^{2-1} \lfloor\underline{2-1}\, a^2}{(ax+b)^2}$$

$$y_3 = \frac{2a^3}{(ax+b)^3} = \frac{(-1)^{3-1} \lfloor\underline{3-1}\, a^3}{(ax+b)^3}$$

$$y_4 = -\frac{2\times 3\, a^4}{(ax+b)^4} = \frac{(-1)^{4-1} \lfloor\underline{4-1}\, a^4}{(ax+b)^4}$$

$$\therefore \boxed{y_n = \frac{(-1)^{n-1}\,\lfloor n-1\;a^n}{(ax+b)^n}}$$ $\left[\begin{array}{l}\because y = \log(x-2) \\ \therefore y_{20} = \dfrac{-\lfloor 19}{(x-2)^{20}}\end{array}\right]$

6) Let $y = x^m$

$\therefore y_1 = m x^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)\,x^{m-3}$

$\therefore y_n = \boxed{m(m-1)(m-2)\ldots(m-\overline{n-1})\,x^{m-n}}$ [when, $m > n$]

$y_{n-1} = m(m-1)(m-2)\ldots(m-\overline{n-2})\,x^{m-\overline{n-1}}$

$y_m = m(m-1)(m-2)\ldots 2.1$ $(n=m)$

$= \lfloor m$

$y_n = 0$ $(n > m)$

7) Let $y = e^{ax}\sin bx$

$\therefore y_1 = a e^{ax}\sin bx + b e^{ax}\cos bx$

$= e^{ax}(a\sin bx + b\cos bx)$

$= e^{ax}(r\cos\theta\sin bx + r\sin\theta\cos bx)$

$= r e^{ax}\sin(bx+\theta) = \sqrt{a^2+b^2}\, e^{ax}\sin(bx+\theta)$

$= (a^2+b^2)^{1/2} e^{ax}\sin(bx+\theta)$

Let

$a = r\cos\theta$

$b = r\sin\theta$

$\therefore r = \sqrt{a^2+b^2}$

$\therefore \theta = \tan^{-1}\frac{b}{a}$

$\therefore y_2 = (a^2+b^2)^{1/2} e^{ax}\{a\sin(bx+\theta) + b\cos(bx+\theta)\}$

$= (a^2+b^2)^{1/2} e^{ax}\, r\sin(bx+2\theta)$

$= (a^2+b^2)^{2/2} e^{ax}\sin(bx+2\theta)$

$\therefore y_3 = (a^2+b^2)^{3/2} e^{ax}\sin(bx+3\theta)$

$\therefore \boxed{y_n = (a^2+b^2)^{n/2} e^{ax}\sin(bx+n\underset{\tan^{-1}\frac{b}{a}}{\theta})}$

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1) If $y = \dfrac{x^3}{x^2-1}$ then find the value of $y_n\ (n>1)$ and hence ~~s~~ show that $y_n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\lfloor n & \text{if } n \text{ is odd} \end{cases}$!

$\Rightarrow$ $y = \dfrac{x^3}{x^2-1}$

$= x + \dfrac{x}{x^2-1}$

$= x + \dfrac{x}{(x+1)(x-1)}$

$= x + \dfrac{1}{2}\left[\dfrac{1}{x-1} + \dfrac{1}{x+1}\right]$

Diff. both sides n times with respect to x we have,

$y_n = 0 + \dfrac{1}{2}\left[\dfrac{(-1)^n \lfloor n}{(x-1)^{n+1}} + \dfrac{(-1)^n \lfloor n}{(x+1)^{n+1}}\right]$

$= \dfrac{1}{2}(-1)^n \lfloor n \left[\dfrac{1}{(x-1)^{n+1}} + \dfrac{1}{(x+1)^{n+1}}\right]$

~~The~~ $\therefore y_n(0) = \dfrac{(-1)^n \lfloor n}{2}\left[\dfrac{1}{(-1)^{n+1}} + \dfrac{1}{(1)^{n+1}}\right]$

$= \dfrac{(-1)^n \lfloor n}{2}\left[-1\,(-1)^n + 1\right] = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\lfloor n, & \text{if } n \text{ is odd} \end{cases}$

2) Liebnitz Theorem

~~$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + {}^nc_3 u_{n-3} v + \dots + {}^nc_n u v_n$~~

Let, u and v be both n times differentiable with respect to x then uv is n times differentiable with respect to x and $(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + {}^nc_3 u_{n-3} v_3 + \dots + {}^nc_n u v_n$

3) if $y = e^{m \sin^{-1} x}$ then show that – i) $(1-x^2)y_2 - xy_1 - m^2 y = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$.

Hence find $y_n(0) = ?$

$\Rightarrow$ given that $y = e^{m\sin^{-1}x}$ ——(i)

Therefore, $y_1 = e^{m\sin^{-1}x} \dfrac{m}{\sqrt{1-x^2}}$ ——(ii)

$= \dfrac{my}{\sqrt{1-x^2}}$

or, $(1-x^2)y_1^2 = m^2y^2$

or, $(1-x^2)2y_1y_2 - 2xy_1^2 = 2m^2yy_1$

or, $(1-x^2)y_2 - xy_1 - m^2y = 0$ ___ (iii)

differentiating of both sides of (iii) n times using Liebnitz theorem we have,

$$\left((1-x^2)y_2\right)_n - (xy_1)_n + (-m^2y)_n = 0$$

or, $\left\{(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n\right\}$
$- \left\{xy_{n+1} + {}^nc_1 y_n\right\} - m^2y_n = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - {}^nc_1y_n - m^2y_n = 0$

or, $(1-x^2)y_{n+2} - y_{n+1}x(2n+1) - y_n\{n(n-1)+n+m^2\} = 0$

or, $(1-x^2)y_{n+2} - y_{n+1}x(2n+1) - y_n(n^2+m^2) = 0$ ___ (iv)

From (i), (ii) and (iii) we have,

$y(0) = 1$; $y_1(0) = \dfrac{my(0)}{\sqrt{1-0}} = m$

and $y_2(0) - m^2 = 0$

or, $y_2(0) = m^2$

Putting $x=0$ and $n=1$ into (iv)

$y_3(0) - (1^2+m^2)y_1(0) = 0$

or, $y_3(0) = m(1^2+m^2)$

Similarly putting $x=0$ and $n=2,3,4\ldots$ Successively into (iv)

$y_4(0) = m^2(2^2+m^2)$

$y_5(0) - (3^2+m^2)y_3(0) = 0$

or, $y_5(0) = m(1^2+m^2)(3^2+m^2)$

$\therefore y_6(0) = m^2(2^2+m^2)(4^2+m^2)$

Therefore, $y_n(0) = \begin{cases} m(1^2+m^2)(3^2+m^2)\ldots(\overline{n-2}^2+m^2), & \text{if } n \text{ is odd} \\ m^2(2^2+m^2)(4^2+m^2)\ldots(\overline{n-2}^2+m^2), & \text{if } n \text{ is even} \end{cases}$

4) If $y = x^{n-1} \log x$, then show that $y_n = \frac{\lfloor n-1}{x}$

⇒ given that $y = x^{n-1} \log x$

differentiating both side with respect to x,

$$y_1 = x^{n-1} \cdot \frac{1}{x} + (n-1) x^{n-2} \log x$$

$$\text{or, } x y_1 = x^{n-1} + (n-1) x^{n-1} \log x$$

$$= x^{n-1} + (n-1) y$$

differentiating both sides $(n-1)$ times with respect to x using Leibnitz rules,

$$(xy_1)_{n-1} = (x^{n-1})_{n-1} + \{(n-1)y\}_{n-1}$$

$$\text{or, } x y_n + {}^{n-1}C_1 (1) y_{n-1} = \lfloor n-1 + (n-1) y_{n-1}$$

$$\text{or, } x y_n + \cancel{(n-1) y_{n-1}} = \lfloor n-1 + \cancel{(n-1) y_{n-1}}$$

$$\text{or, } y_n = \frac{\lfloor n-1}{x}$$

5) If $I_n = \frac{d^n}{dx^n}(x^n \log x)$ then show that $I_n = n I_{n-1} + \lfloor n-1$ and hence show that $I_n = \lfloor n \left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$

⇒ Given that, $I_n = \frac{d^n}{dx^n}(x^n \log x)$ — (i)

$$\text{Therefore, } I_n = \frac{d^{n-1}}{dx^{n-1}} \left(\frac{d}{dx}(x^n \log x)\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}} \left(x^{n-1} + n x^{n-1} \log x\right)$$

$$= \lfloor n-1 + n \frac{d^{n-1}}{dx^{n-1}} \left(x^{n-1} \log x\right)$$

$$\therefore I_n = \lfloor n-1 + n I_{n-1} \quad [\text{by (i)}]$$

$$\text{Therefore, } \frac{I_n}{\lfloor n} = \frac{I_{n-1}}{\lfloor n-1} + \frac{1}{n} \quad \text{— (ii)}$$

Putting $n = n, n-1, n-2, \dots, 4, 3, 2$ Succesively into both sides of (ii) and adding columnwise we have,

$$\frac{I_n}{\lfloor n} = \cancel{\frac{I_{n-1}}{\lfloor n-1}} + \frac{1}{n}$$

$$\cancel{\frac{I_{n-1}}{\lfloor n-1}} = \cancel{\frac{I_{n-2}}{\lfloor n-2}} + \frac{1}{n-1}$$

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$$\cancel{\frac{I_4}{\lfloor 4}} = \cancel{\frac{I_3}{\lfloor 3}} + \frac{1}{4}$$

$$\cancel{\frac{I_3}{\lfloor 3}} = \cancel{\frac{I_2}{\lfloor 2}} + \frac{1}{3}$$

$$\cancel{\frac{I_2}{\lfloor 2}} = \frac{I_1}{\lfloor 1} + \frac{1}{2}$$

$$\frac{I_n}{\lfloor n} = \frac{I_1}{\lfloor 1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n-1} + \frac{1}{n}$$

again from (i) $I_1 = \frac{d}{dx}(x \log x) = x \cdot \frac{1}{x} + \log x$

$$= 1 + \log x$$

$$\therefore \frac{I_n}{\lfloor n} = \log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

or, $I_n = \lfloor n \left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} \right)$

Envelope

Definition:– If a curve touches every member of a family of curves then the curve is called the Envelope of the family.

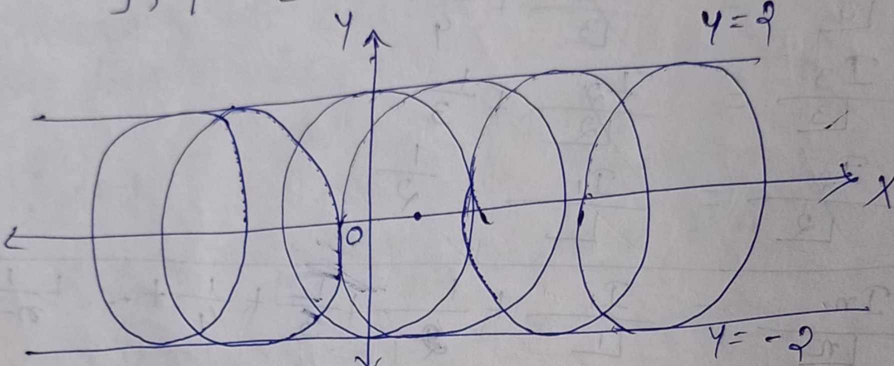
For example consider the family of the circles,

$$(x-\alpha)^2 + y^2 = 4 \quad \text{—(i)}, \text{ where } \alpha \text{ is a parameter}$$

Since, $y = 2$ touches every member of the family (i)

Therefore, $y = 2$ is an envelope of the family (i).

Similarly, $y = -2$ is also an envelope of the family (i).



1) Find the envelope of the family of the straight lines represented by $y = mx + \sqrt{a^2m^2 + b^2}$, where m is a parameter, (a, b constant)

⇒ the given equation can be written as

$$(y - mx)^2 = a^2m^2 + b^2$$

or, $(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0$ —(i)

(i) is quadratic in m,

Therefore, the envelope is obtained by equating the discriminant of (i) to zero,

$$\therefore (-2xy)^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

or, $x^2y^2 - x^2y^2 - a^2b^2 + a^2y^2 + b^2x^2 = 0$

or, $a^2y^2 + b^2x^2 = a^2b^2$

or, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

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2) Find the envelope of the family of straight line $y = mx + \frac{a}{m}$, $m \neq 0$ being Parameter.

$\Rightarrow$ The equation of the family can be written as,

$$xm^2 - ym + a = 0 \quad \text{--- (i)}$$

(i) is quadratic in the parameter m.

Therefore, the envelop of the given family is obtained by equating the discriminant of (i) to zero,

$$(-y)^2 - 4ax = 0$$

or, $y^2 = 4ax$.

3) Find the envelope of the family of straight lines represented by $y = \frac{x}{m} + m^2$, where m is a parameter.

$\Rightarrow$ The equation of the given family can be written as,

$$m^3 - ym + x = 0 \quad \text{--- (i)}$$

differenting both sides of (i) partially with respect to the parameter m we have,

$$3m^2 - y = 0 \quad \text{--- (ii)}$$

The envelope is obtained by eleminating the parameter m between (i) and (ii)

from (ii), $m^2 = \frac{y}{3}$ --- (iii)

Therefore, from (i),

$$m(m^2 - y) = -x$$

or, $m^2(m^2 - y)^2 = x^2$

or, $\frac{y}{3}\left(\frac{y}{3} - y\right)^2 = x^2$ [by (iii)]

or, $\frac{y}{3} \times \frac{4y^2}{9} = x^2$

or, $4y^3 = 27x^2$

This is the required envelope.

3) Find the envelope of the family of curves $y = 2mx + m^4$, m being parameter.

=> The given equation can be written as,

$$m^4 + 2mx - y = 0 \quad \text{—(i)}$$

differentiating both sides of (i) partially with respect to m we get

$$4m^3 + 2x = 0 \quad \text{—(ii)}$$

From (ii),

$$m^3 = \frac{-2x}{4} = -\frac{x}{2}$$

Therefore from (i),

$$m(m^3 + 2x) = y$$

or, $m^3(m^3 + 2x)^3 = y^3$

or, $-\frac{x}{2}\left(-\frac{x}{2} + 2x\right)^3 = y^3$

or, $-\frac{x}{2} \times \frac{27x^3}{8} = y^3$

or, $27x^4 + 16y^3 = 0$

4) Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where a and b are parameters connected by the relation $a^n + b^n = c^n$, c being a constant.

=> The equation of the family,

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{—(i)}$$

and the relations between the parameters,

$$a^n + b^n = c^n \quad \text{—(ii)}$$

differentiating of both sides of (i) with respect to a,

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0$$

or, $\frac{db}{da} = -\frac{xb^2}{a^2y} \quad \text{—(iii)}$

differentiating both sides of (ii) with respect to a,

$$na^{n-1} + nb^{n-1}\frac{db}{da} = 0$$

$$\text{or, } \frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \quad \text{--- (iv)}$$

From (iii) and (iv),

$$\cancel{x = \lambda a^{n+1}, \; y =}$$

$$\cancel{= -\frac{a^2 x}{}} \quad -\frac{b^2 x}{a^2 y} = -\frac{a^{n-1}}{b^{n-1}}$$

$$\text{or, } \frac{x}{a^{n+1}} = \frac{y}{b^{n+1}} = \lambda \text{ (say)}$$

$$x = \lambda a^{n+1}, \; y = \lambda b^{n+1} \quad \text{--- (v)}$$

Therefore, from (i),

$$\frac{\lambda a^{n+1}}{a} + \frac{\lambda b^{n+1}}{b} = 1$$

$$\text{or, } \lambda(a^n + b^n) = 1$$

$$\text{or, } \lambda c^n = 1 \quad [\text{by (ii)}]$$

$$\text{or, } \lambda = \frac{1}{c^n}$$

Therefore from (v),

$$x = \frac{a^{n+1}}{c^n}, \quad y = \frac{b^{n+1}}{c^n}$$

$$\text{or, } a = (xc^n)^{\frac{1}{n+1}}, \quad \text{or, } b = (yc^n)^{\frac{1}{n+1}}$$

Putting the values of the parameters a and b into (ii),

$$(c^n x)^{\frac{n}{n+1}} + (yc^n)^{\frac{n}{n+1}} = c^n$$

$$\text{or, } x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = \frac{c^n}{c^{\frac{n^2}{n+1}}} = c^{n - \frac{n^2}{n+1}} = c^{\frac{n}{n+1}}$$

$$\text{or, } x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$$

5) Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are parameters, connected by $a^2 b^3 = c^5$, c being a constant.

$\Rightarrow$ the given equations are,

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{—(i)}$$

and,

$$a^2 b^3 = c^5 \quad \text{—(ii)}$$

differentiating both sides of (i) with respect to a,

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0$$

$$\text{or, } \frac{db}{da} = -\frac{x b^2}{a^2 y} \quad \text{—(iii)}$$

differentiating both sides of (ii) with respect to a,

$$2ab^3 + 3b^2a^2\frac{db}{da} = 0$$

$$\text{or, } \frac{db}{da} = -\frac{2b}{3a} \quad \text{—(iv)}$$

from (iii) and (iv),

$$\cancel{+}\frac{b^{\cancel{2}} x}{a^{\cancel{2}} y} = \cancel{+}\frac{2\cancel{b}}{3\cancel{a}}$$

$$\text{or, } \frac{x}{2a} = \frac{y}{3b} = \lambda \text{ (say)}$$

$$x = 2a\lambda, \quad y = 3b\lambda \quad \text{—(v)}$$

from (i) $\Rightarrow$

$$\frac{2a\lambda}{a} + \frac{3b\lambda}{b} = 1$$

or, $5\lambda = 1$

or, $\lambda = \frac{1}{5}$

From (v) $\Rightarrow$

$$x = \frac{2a}{5}, \quad y = \frac{3b}{5}$$

or, $a = \frac{5x}{2}$, $b = \frac{5y}{3}$

~~from (ii) $\Rightarrow$~~ putting the values of a and b in (ii),

$$\left(\frac{5x}{2}\right)^2 \left(\frac{5y}{3}\right)^3 = c^5$$

or, $5^5 x^2 y^3 = 2^2 3^3 c^5$.

6) Find the envelope of the family $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where a and b are parameters connected by $ab = k^2$, k being a constant.

$\Rightarrow$ The given equations are,

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \quad \text{—(i)}$$

and $$ab = k^2 \quad \text{—(ii)}$$

differentiating (i) with respect to a,

$$\sqrt{x}\, a^{-1/2} + \sqrt{y}\, b^{-1/2} = 1$$

or, $$-\frac{1}{2}\sqrt{x}\, a^{-3/2} - \frac{1}{2}\sqrt{y}\, b^{-3/2} \frac{db}{da} = 0$$

or, $$\frac{db}{da} = -\frac{\sqrt{x}\, a^{-3/2}}{\sqrt{y}\, b^{-3/2}}$$

$$= -\frac{\sqrt{x}\, b^{3/2}}{\sqrt{y}\, a^{3/2}} \quad \text{—(iii)}$$

differentiating (ii) with respect to a,

$$b + a\frac{db}{da} = 0$$

or, $$\frac{db}{da} = -\frac{b}{a} \quad \text{—(iv)}$$

from (iii) and (iv),

$$-\frac{\sqrt{x}\, b^{3/2}}{\sqrt{y}\, a^{3/2}} = -\frac{b}{a}$$

or, $$\frac{\sqrt{x}}{\sqrt{a}} = \frac{\sqrt{y}}{\sqrt{b}} = \lambda \text{ (say)}$$

$$\sqrt{x} = \lambda\sqrt{a}, \quad \sqrt{y} = \lambda\sqrt{b} \quad \text{—(v)}$$

from (i),

$$\frac{\lambda\sqrt{a}}{\sqrt{a}} + \frac{\lambda\sqrt{b}}{\sqrt{b}} = 1$$

or, $2\lambda = 1$

or, $\lambda = \frac{1}{2}$

from (v),

$$\sqrt{x} = \frac{\sqrt{a}}{2} \qquad \sqrt{y} = \frac{\sqrt{b}}{2}$$

or, $b = 4y$

or, $a = \cancel{2}\, 4x$

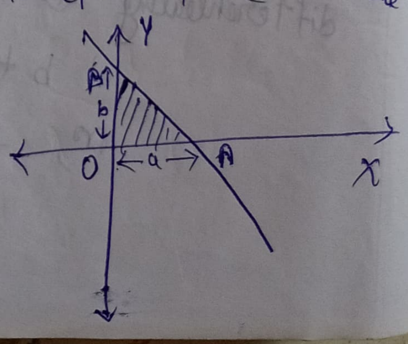
Putting the values of a and b into (ii),

$$(4x)(4y) = k^2$$

or, $16xy = k^2$

This is the required envelope.

7) Find the envelope of the family of straight lines which together with the line segment intercepted by the coordinate axis ~~form~~ form triangle of equal area.



⇒ Let the equation of the family of straight lines be, $\frac{x}{a} + \frac{y}{b} = 1$ —(i)

By the given condition,

$$\frac{1}{2}ab = \text{Cons}$$

or, $ab = \text{Cons} = c^2$ (say)

or, $ab = c^2$ —— (ii)

differentiating both sides of (i) with respect to a,

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0$$

or, $\dfrac{db}{da} = -\dfrac{xb^2}{ya^2}$ —— (iii)

differentiating both sides of (ii) with respect to a,

$$b + a\frac{db}{da} = 0$$

or, $\dfrac{db}{da} = -\dfrac{b}{a}$ —— (iv).

from (iii) and (iv),

$$-\frac{x\cancel{b^2}}{y a^{\cancel{2}}} = -\frac{\cancel{b}}{\cancel{a}}$$

or, $\dfrac{x}{a} = \dfrac{y}{b} = \lambda$ (say)

$x = a\lambda$, $y = b\lambda$ —— (v)

from (i),

$\lambda + \lambda = 1$

or, $\lambda = \frac{1}{2}$

from (v),

$x = \dfrac{a}{2}$, $y = \dfrac{b}{2}$

or, $a = 2x$, or, $b = 2y$

Putting the value of a and b into (ii),

$$(2x)(2y) = c^2$$

or, $4xy = c^2$

This is the required envelope.

8) Find the envelope of the family of ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{(a-\alpha)^2}=1$ where α is a parameter.

$\Rightarrow$ Let, $\beta = a-\alpha$

Since α is a parameter.

Therefore β is also a parameter.

Therefore, the equation of the given family is

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1 \quad \text{—(i)}$$, where the parameters α and β are connected by $\alpha+\beta=a$ —(ii)

differentiating (i) with respect to α

$$x^2\alpha^{-2}+y^2\beta^{-2}=1$$

or, $$-2x^2\alpha^{-3}-2y^2\beta^{-3}\frac{d\beta}{d\alpha}=0$$

or, $$\frac{x^2}{\alpha^3}+\frac{y^2}{\beta^3}\frac{d\beta}{d\alpha}=0$$

or, $$\frac{d\beta}{d\alpha}=-\frac{x^2\beta^3}{\alpha^3y^2} \quad \text{—(iii)}$$

differentiating both sides of (ii) with respecte to α,

$$1+\frac{d\beta}{d\alpha}=0$$

or, $$\frac{d\beta}{d\alpha}=-1 \quad \text{—(iv)}$$

$\therefore$ from (iii) and (iv),

$$\frac{x^2\beta^3}{\alpha^3y^2}=1$$

or, $$\frac{x^2}{\alpha^3}=\frac{y^2}{\beta^3}=\lambda \text{ (Say)}$$

$$x^2=\alpha^3\lambda,\quad y^2=\beta^3\lambda \quad \text{—(v)}$$

from (i) $\dfrac{\alpha^3\lambda}{\alpha^2} + \dfrac{\beta^3\lambda}{\beta^2} = 1$

or, $\alpha\lambda + \beta\lambda = 1$

or, $\lambda = \dfrac{1}{\alpha+\beta}$

from (v)

$\therefore x^2 = \dfrac{\alpha^3}{\alpha+\beta}$, $y^2 = \dfrac{\beta^3}{\alpha+\beta}$

$\cancel{\alpha+\beta =}$

$\alpha+\beta = \dfrac{\alpha^3}{x^2}$ | $\alpha+\beta = \dfrac{\beta^3}{y^2}$

$\cancel{\dfrac{x}{\alpha^2} = \dfrac{\beta^2}{y}}$ $\quad \dfrac{\alpha^3}{x^2} = \dfrac{\beta^3}{y^2} = a$

or, $\alpha^3 = ax^2$, $\beta^3 = y^2a$

or, $\alpha = (ax^2)^{1/3}$, $\beta = (y^2a)^{1/3}$

$\therefore$ ~~from (ii)~~ putting the values of α and β into (ii)

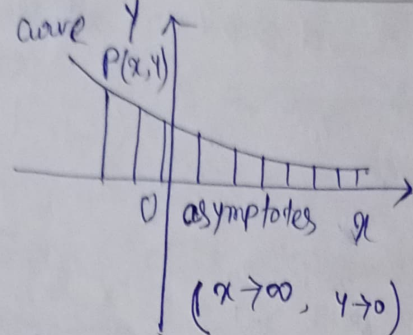
$(ax^2)^{1/3} + (ay^2)^{1/3} = a$

or, $a^{1/3}\left(x^{2/3} + y^{2/3}\right) = a$

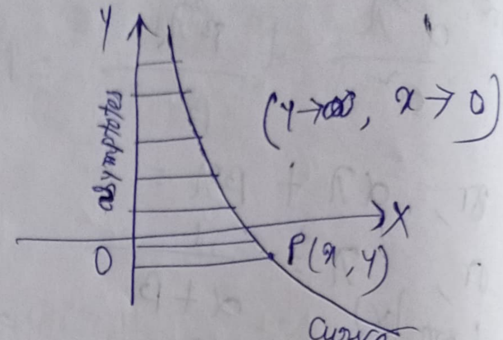
or, $x^{2/3} + y^{2/3} = a^{2/3}$

This is the required envelope.

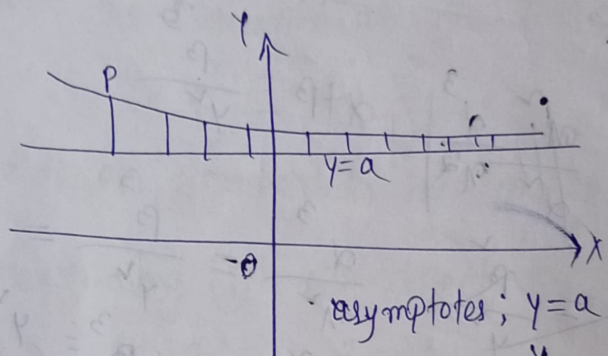
Asymptotes



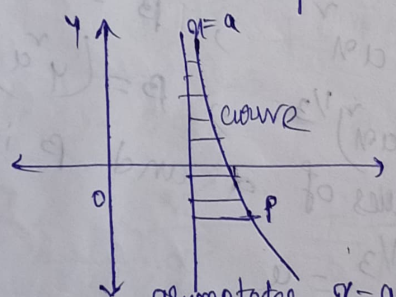
x-axis is asymptotes



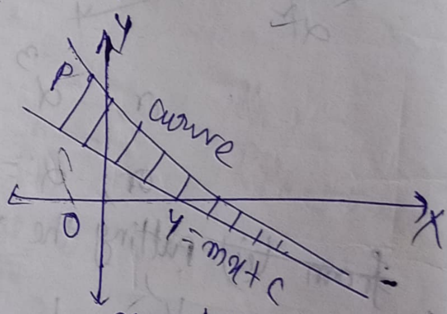
y-axis is asymptotes



asymptotes; $y=a$



asymptotes, $x=a$



asymptotes, $y=mx+c$

Asymptotes

- Parallel (either axis or parallel of the axis)
  - horizontal (either x axis or parallel to x axis)
  - vertical (either y-axis or parallel to y axis)
- Oblique (not parallel of any axis)

1) Find the parallel asymptotes of the curve $x^2y^2 - a^2(x^2+y^2) - a^3(x+y) + a^4 = 0$

⇒ Horizontal asymptotes :-

the highest power of x in the given equation is 2, and the cooficient of highest power of x (x^2) is $y^2 - a^2$.

The horizental asymptotes is obtained by equating the

cooficient of highest power of x to zero,
Therefore,

$$y^2 - a^2 = 0$$

or, $y = \pm a$

Vertical asymptotes :-

The highest power of y is 2 and the cooficient of highest power of that y (y^2) is, $x^2 - a^2$

vertical asymptotes are obtained by equating the cooficient of highest power of y to zero.

Therefore, $x^2 - a^2 = 0$

or, $x = \pm a$

2) Find the parallal asymptotes of the curve, $3x^3 + xy^2 + 2xy + 5 = 0$

⇒ Horigental asymptotes :-

Since the cooficient of highest power of x is constant. Therefore, the given curve has no horigental asymptotes.

vertical asymptotes :-

Here the cooficient of highest power of y (y^2) is x. Therefore, $x = 0$ is a vertical asymptotes.

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3) Find the assymptotes of the curve $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$;

⇒ Parrallal assymptotes :-

Since, the cooficients of highest power of x and y are both constant the curve has no parallal asymptotes.

Oblique asymptotes :-

The given equation of the curve —

$$\underbrace{y^3 - x^2y - 2xy^2 + 2x^3}_{\text{3rd degree terms}} \underbrace{- 7xy + 3y^2 + 2x^2}_{\text{2nd degree terms}} + \underbrace{2x + 2y + 1}_{\text{1st degree terms}} = 0 \quad \text{(i)}$$

Let $y = mx + c$ be oblique asymptotes.

$\phi_3(m) = m^3 - m - 2m^2 + 2$, $\phi_3'(m) = 3m^2 - 1 - 4m$

$\phi_2(m) = -7m + 3m^2 + 2$

$\phi_1(m) = 2 + 2m$

Now, $\phi_3(m) = 0$ gives,

$$m^3 - m - 2m^2 + 2 = 0$$

or, $m(m^2 - 1) - 2(m^2 - 1) = 0$

or, $(m^2 - 1)(m - 2) = 0$.

$\therefore m = 1, -1, 2$.

(i) When $m = 1$ we have,

$$c = -\frac{\phi_2(1)}{\phi_3'(1)} \quad \left[\because c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}\right]$$

$$= -\frac{-2}{-2} = -1$$

$\therefore y = x - 1$ is an oblique asymptotes.

ii) When $m = -1$ we have,

$$c = -\frac{\phi_2(-1)}{\phi_3'(-1)}$$

$$= -\frac{7+3+2}{3-1+4} = -\frac{12}{6} = -2$$

$\therefore y = -x - 2$ that is, $x + y + 2 = 0$ is another oblique asymptotes.

iii) When $m = 2$ we have,

$$c = -\frac{\phi_2(2)}{\phi_3'(2)}$$

$$= -\frac{-14+12+2}{12-1-8} = 0$$

$\therefore y = 2x$ is another oblique asymptotes.

Note: If $c = \frac{-\phi_{n-1}(m)}{\phi_n'(m)} \left(\frac{0}{0}\right)$ then 'c' is given by

$$\frac{c^2}{2}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0.$$

4) Find the asymptotes of the following curve $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$.

⇒ Parrallal asymptotes :-

Since, the ~~highest power of x~~ cooficient of highest power of x and y are both constant, the curve has no parallal asymptotes.

Oblique asymptotes :-

The given equation of the curve —

$$\underbrace{4x^3 - 3xy^2 - y^3}_{\text{3rd degree terms}} + \underbrace{2x^2 - xy - y^2}_{\text{2nd degree terms}} - 1 = 0$$

Let $y = mx + c$ be oblique asymptotes.

we have, $\phi_3(m) = 4 - 3m^2 - m^3$, $\phi_3'(m) = -6m - 3m^2$, $\phi_3''(m) = -6 - 6m$

$\phi_2(m) = 2 - m - m^2$, $\phi_2'(m) = -1 - 2m$

$\phi_1(m) = 0$

Now, $\phi_3(m) = 0$ gives,

$4 - 3m^2 - m^3 = 0$

or, $m^3 + 3m^2 - 4 = 0$

or, $m^3 - m^2 + 4m^2 - 4m + 4m - 4 = 0$

or, $m^2(m-1) + 4m(m-1) + 4(m-1) = 0$

or, $(m-1)(m^2 + 4m + 4) = 0$

or, $(m-1)(m+2)^2 = 0$

$\therefore m = 1, -2, -2$

i) when $m = 1$, $c = -\frac{\phi_2(1)}{\phi_3'(1)} = -\frac{2-1-1}{-6-3} = 0$

$\therefore y = x$ is an oblique asymptotes.

ii) when $m = -2$, $c = -\frac{\phi_2(-2)}{\phi_3'(-2)} = -\frac{2+2-4}{0} = \frac{0}{0}$

Therefore, we have,

$$\frac{c^2}{2}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

or, $\frac{c^2}{2}\phi_3''(-2) + c\phi_2'(-2) + \phi(-2) = 0$

or, $\frac{c^2}{2} 6 + c\cdot 3 + 0 = 0$

or, $c^2 + c = 0$

or, $c(c+1) = 0$

or, $c = 0,\ c = -1$

$\therefore$ $y = -2x$ and $y = -2x - 1$ are other asymptotes.

asymptotes for Polar curve

1) Find the asymptotes of the curve $r\theta = a$.

$\Rightarrow$ From the given equation of the curve we have,

$$r\theta = a$$

or, $\frac{1}{r} = \frac{\theta}{a}$

or, $u = \frac{\theta}{a}$ $\left[\text{Let } \frac{1}{r} = u\right]$ —— (i)

now, $u = 0$ gives,

$\frac{\theta}{a} = 0$

or, $\theta = 0 (= \alpha)$

from (i), $\frac{du}{d\theta} = \frac{1}{a}$

$\therefore p = \left.\frac{d\theta}{du}\right]_{\theta=\alpha} = a\Big]_{\theta=\alpha} = a$

$\therefore$ The asymptotes of the curve is $p = r\sin(\theta - \alpha)$.

~~$\Rightarrow p = a\sin\theta$.~~

$\Rightarrow a = r\sin\theta$

2) Find the asymptotes of the curve $r \cos\theta = a \sin\theta$

$\Rightarrow$ From the equation of the given curve,

$$r = a \tan\theta$$

$$\text{or, } u = \frac{1}{r} = \frac{1}{a}\cot\theta \quad \text{(i)}$$

$u=0$ gives,

$$\cot\theta = 0$$

$$\text{or, } \theta = \frac{\pi}{2} \ (=\alpha)$$

from (i), $\frac{du}{d\theta} = -\frac{1}{a}\operatorname{cosec}^2\theta$

Now, $p = \left.\frac{d\theta}{du}\right|_{\theta=\alpha} = -a\sin^2\theta\Big|_{\theta=\frac{\pi}{2}} = -a$

$\therefore$ The asymptotes is ~~$p = p\sin(\theta-\alpha)$~~ $p = r\sin(\theta-\alpha)$

$$\Rightarrow -a = r\sin\left(\theta - \frac{\pi}{2}\right)$$

~~$\Rightarrow -a = r \sin$~~

~~$\Rightarrow r = -a\sin\left(\theta-\frac{\pi}{2}\right) = a\cos\theta$~~.

3) Find the asymptotes of the curve $r = \frac{a}{1-\cos\theta}$, if any.

$\Rightarrow$ From the equation of the given curve,

$$\frac{1}{r} = \frac{1-\cos\theta}{a}$$

$$\text{or, } u = \frac{1-\cos\theta}{a} \quad \text{(i)}$$

$u=0$ gives,

$$1-\cos\theta = 0$$

$$\text{or, } \cos\theta = 1$$

$$\text{or, } \theta = 0 \ (=\alpha)$$

from (i), $\frac{du}{d\theta} = -\frac{\sin\theta}{a}$

$\therefore p = \left.\frac{d\theta}{du}\right|_{\theta=0} = \frac{-a}{\sin\theta} = \infty$

$\therefore$ The curve has no asymptotes.

4) Find the asymptotes of the curve $r = \frac{a}{\frac{1}{2} - \cos\theta}$.

$\Rightarrow$ The given equation is written as,

$$\frac{1}{r} = \frac{\frac{1}{2} - \cos\theta}{a}$$

or, $u = \frac{\frac{1}{2} - \cos\theta}{a}$ $\left[u = \frac{1}{r}\right]$ —(i)

$u = 0$ gives,

$\frac{1}{2} - \cos\theta = 0$

or, $\cos\theta = \frac{1}{2}$

or, $\theta = \frac{\pi}{3}$ $(=\alpha)$

from (i),

$\frac{du}{d\theta} = \frac{\sin\theta}{a}$

$\therefore P = \frac{d\theta}{du}\Big]_{\theta=\frac{\pi}{3}} = \frac{a}{\sin\theta}\Big]_{\theta=\frac{\pi}{3}} = \frac{a}{\sin\frac{\pi}{3}}$

$= \frac{2a}{\sqrt{3}}$

$\therefore$ The asymptotes is $p = \cancel{r} \sin(\theta - \alpha)$

$\frac{2a}{\sqrt{3}} = \cancel{\frac{2a}{\sqrt{3}}}\, r \sin\left(\theta - \frac{\pi}{3}\right)$

Asymptotes for parametric curve

1) Find the asymptotes of the curve which parametric equation is $x = \frac{2t}{t^2-1}$, $y = \frac{(t+1)^2}{t^2}$.

$\Rightarrow$ The values of t for which x or y are undefined are $t = 0, 1, -1$.

i) When $t \to 0$, $x \to 0$ and $y \to \infty$

$\therefore x = 0$ is an asymptotes.

ii) When $t \to 1$, $x \to \infty$ and $y \to 4$.

$\therefore y = 4$ is an asymptotes.

iii) When $t \to -1$, $x \to \infty$ and $y \to 0$

$\therefore y = 0$ is an asymptotes.

2) Find the asymptotes of the following parametric curve —

$$x = \frac{t^2}{1+t^3}, \quad y = \frac{t^2+2}{1+t}.$$

$\Rightarrow$ $1+t^3 = 0$ gives,

$t = -1, -\omega, -\omega^2$

and $1+t = 0$ gives,

$t = -1$

$\therefore$ The distinct real value of t for which x or y are undefine is $t = -1$.

Now, ~~for~~, $t \to -1$ we have $x \to \infty$ and $y \to \infty$

$\therefore$ The curve has an oblique asymptotes.

Let $y = mx + c$ be an oblique asymptotes.

$$\therefore m = \lim_{t \to -1} \frac{y}{x} = \lim_{t \to -1} \frac{(t^2+2)(1+t^3)}{(1+t)t^2} =$$

$$= \lim_{t \to -1} \frac{(t^2+2)(t^2-t+1)}{t^2} = \frac{(1+2)(1+1+1)}{1} = 9$$

$$\therefore c = \lim_{t \to -1} (y - mx) = \lim_{t \to -1} \left(\frac{t^2+2}{1+t} - 9\frac{t^2}{1+t^3} \right)$$

$$= \lim_{t \to -1} \frac{(t^2+2)(t^2-t+1) - 9t^2}{(1+t^3)}$$

$$= \lim_{t \to -1} \frac{t^4 + 2t^2 - t^3 - 2t + t^2 + 2 - 9t^2}{1+t^3} = \lim_{t \to -1} \frac{t^4 - t^3 - 6t^2 - 2t + 2}{1+t^3} \left(\frac{0}{0}\right)$$

$$= \lim_{t \to -1} \frac{4t^3 - 3t^2 - 12t - 2}{3t^2} = \frac{-4-3+12-2}{3} = 1$$

$\therefore$ The asymptote is $y = 9x + 1$

Date - 14/7/18

L' Hospitals rule

Statement:- If two functions $f(x)$ and $g(x)$ are such that $f^n(x)$ and $g^n(x)$ exists in some deleted neighbourhood of 'a' where $g^n(x) \neq 0$, $\lim_{x \to a} f^r(x) = \lim_{x \to a} g^r(x) = 0$ $(r = 1, 2, 3, \ldots n-1)$ and $\lim_{x \to a} \frac{f^n(x)}{g^n(x)}$ exists then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}$

1) i) $\frac{0}{0}$ form :- evaluate the limit $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$.

$$\Rightarrow \lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x + xe^x - \frac{1}{1+x}}{2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x + e^x + xe^x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{2e^0 + 0 + \frac{1}{(1)^2}}{2} = \cancel{\frac{3}{2}}\ \frac{3}{2}$$

2) If $\lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3}$ exists a finite number, find the value of a and evaluate the limit.

$$\Rightarrow \lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\cos 2x + a\cos x}{3x^2} \quad \left(\frac{2+a}{0}\right)$$

[Since, the limit exists $\therefore 2 + a = 0$ or, $a = -2$]

$$= \lim_{x \to 0} \frac{2\cos 2x - 2\cos x}{3x^2}$$

$$= \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-6}{6} = -1.$$

3) Find the value of a and b so that $\lim_{x\to 0} \frac{a\sin 2x - b\sin x}{x^3} = 1$

$\Rightarrow \lim_{x\to 0} \frac{a\sin 2x - b\sin x}{x^3} \left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{2a\cos 2x - b\cos x}{3x^2} \left(\frac{2a-b}{0}\right)$ [Since the limit exists, $\therefore 2a-b=0$ $\therefore 2a=b$]

$= \lim_{x\to 0} \frac{2a\cos 2x - 2a\cos x}{3x^2}$

$= 2a \lim_{x\to 0} \frac{-2\sin 2x + \sin x}{6x} \left(\frac{0}{0}\right)$

$= 2a \lim_{x\to 0} \frac{-4\cos 2x + \cos x}{6}$

$= -\frac{2a}{2} = -a$

By ~~From~~ the given condition,

$-a = 1$

or, $a = -1$

$\therefore b = -2$

ii) ~~Infinity~~ $\frac{\infty}{\infty}$ form:- evaluate the limit $\lim_{x\to 0} \frac{\log \sin 2x}{\log \sin x}$

$\Rightarrow \lim_{x\to 0} \frac{\log \sin 2x}{\log \sin x} \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to 0} \frac{\frac{2\cos 2x}{\sin 2x}}{\frac{\cos x}{\sin x}} \left(\frac{\infty}{\infty}\right) = 2\lim_{x\to 0} \frac{\tan x}{\tan 2x} \left(\frac{0}{0}\right)$

$= \cancel{2} \lim_{x\to 0} \frac{\sec^2 x}{\cancel{2}\sec^2 2x} = \frac{1}{1} = 1$

iii) $\infty - \infty$ form:-

1) Evaluate limit $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

$\Rightarrow \lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) (\infty - \infty)$

$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{2\sin x \cos x - 2x}{2x\sin^2 x + 2x^2 \sin x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin 2x - 2x}{2x\sin^2 x + x^2 \sin 2x} \quad \left(\frac{0}{0}\right) \qquad \left(-\frac{1}{3} \text{ Ans}\right)$$

$$\cancel{= \lim_{x \to 0} \frac{2\sin 2x \cos 2x - 2}{2\sin^2 x + 4x\sin x \cos x + 2x\sin 2x + 2x^2 \sin 2x \cos 2x}}$$

$$= \lim_{x \to 0} \frac{2\cos 2x - 2}{2\sin^2 x + 2x\sin x\cos x + 2x \sin 2x + 2x^2 \cos 2x} = \lim_{x \to 0} \frac{\cos 2x - 1}{\sin^2 x + x^2\cos 2x + 2x}$$

$$= \lim_{x \to 0} \frac{-2\sin 2x}{2\sin x \cos x + 2x\cos 2x - 2x^2 \sin 2x + 2\sin 2x + 4x\cos 2x} = \lim_{x \to 0} \left(\frac{0}{0}\right) \frac{-2\sin 2x}{3\sin 2x + 6x\cos x - 2x^2 \sin 2x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-4\cos 2x}{6\cos 2x + 6\cos x - 6x\sin x - 4x\sin 2x - 4x^2 \cos 2x} = \frac{-4}{6+6} = -\frac{4}{12} = -\frac{1}{3}.$$

2) evaluate limit $\lim_{x \to \infty} \left\{ x - x^2 \log\left(1 + \frac{1}{x}\right) \right\}$ | Let $x = \frac{1}{u}$, When $x \to \infty$, $u \to 0$

$\Rightarrow \lim_{x \to \infty} \left\{ x - x^2 \log\left(1 + \frac{1}{x}\right) \right\}$

$$= \lim_{u \to 0} \left\{ \frac{1}{u} - \frac{1}{u^2} \log(1+u) \right\} (\infty - \infty)$$

$$= \lim_{u \to 0} \frac{u - \log(1+u)}{u^2} \left(\frac{0}{0}\right)$$

$$= \lim_{u \to 0} \frac{1 - \frac{1}{1+u}}{2u} \left(\frac{0}{0}\right)$$

$$= \lim_{u \to 0} \frac{1}{(1+u)^2 2} = \frac{1}{2}$$

iv) 1^∞ form :-

1) evaluate the limit $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

$\Rightarrow$ Let, $L = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

$$\text{or, } \log L = \lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\sin x}{x}\right)$$

$$= \lim_{x \to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{x}{\sin x} \cdot \frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^2 \sin x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{4x\sin x + 2x^2\cos x}$$

$$= \lim_{x\to 0} \frac{-x\sin x}{4x\sin x + 2x^2\cos x} \left(\frac{0}{0}\right)$$

$$= \cancel{\lim_{x\to 0} - \frac{\sin x + x\cos x}{4\sin x + 4x\cos x + 4x\cos x - 2x^2\sin x}} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x}{4\sin x + 2x\cos x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-\cos x}{4\cos x + 2\cos x - 2x\sin x} = -\frac{1}{6}$$

$$\therefore L = e^{-1/6}$$

0^0 form :-

1) evaluate $\lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$.

$\Rightarrow$ Let, $L = \lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$

or, $\log L = \lim_{x\to 1} \frac{1}{\log(1-x)} \log(1-x^2) \left(\frac{\alpha}{\alpha}\right)$

$$= \lim_{x\to 1} \frac{(-2x)(1-x)}{(1-x^2)(-1)} = \lim_{x\to 1} \frac{2x}{1+x} = \frac{2}{2} = 1$$

$\therefore \log L = 1$

$\therefore L = e$

$0.\infty$ form :-

1) evaluate $\lim_{x\to\infty} x\tan\frac{1}{x}$.

Let, $\frac{1}{x} = u$

when $x\to\infty$

$u\to 0$

$\Rightarrow \lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty} \frac{\tan\frac{1}{x}}{\frac{1}{x}}$

$$= \lim_{u\to 0} \frac{\tan u}{u} \left(\frac{0}{0}\right) = \lim_{u\to 0} \frac{\sec^2 u}{1} = 1$$

2) evaluate $\lim_{x \to 0} (\sin x)^{2\tan x}$

$\Rightarrow L = \lim_{x \to 0} (\sin x)^{2\tan x}$

$\log L = \lim_{x \to 0} 2\tan x \log \sin x$

$= 2 \lim_{x \to 0} \frac{\sin x \log \sin x}{\cos x}$

$= 2 \lim_{x \to 0} \frac{\log \sin x}{\cot x} \left(\frac{0}{0}\right)$

$= 2 \lim_{x \to 0} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$

$= -2 \lim_{x \to 0} \frac{\cos x}{\operatorname{cosec} x} = -2 \lim_{x \to 0} \sin x \cos x$

$= 0$

$\therefore L = e^0 = 1$

Hyperbolic function

$\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x$

$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2i} \times i = i \sin x$

$\therefore \tanh(ix) = \frac{\sinh(ix)}{\cosh(ix)} = i \tan x$

$\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cosh(x)$

$\sin(ix) = \frac{e^{-x} - e^{x}}{2i} = -\frac{e^{x} - e^{-x}}{2i} = -\frac{1}{i} \sinh(x)$

$\cos x + i \sin x = e^{ix}$.

$= i \sinh(x)$

general value :-

$\text{Log } z = 2n\pi i + \log z$, where n is integer and z is complex number. ($\text{Log} \Rightarrow$ general, $\log \Rightarrow$ principle)

$\text{Sin } z = 2n\pi i + \sin z$

$\text{Cos } z = 2n\pi i + \cos z$

1) Express $\sin(x+iy)$ in the form $A+iB$.

$\Rightarrow$ $\sin(x+iy) = \sin x \cos(iy) + \cos(x) \sin(iy)$

$= \sin x \cosh y + i \cos x \sinh y$

$= A + iB$ (where $A = \sin x \cosh y$ and $B = \cos x \sinh y$)

2) Express $\tan^{-1}(x+iy)$ in the form $A+iB$.

$\Rightarrow$ Let, $\tan^{-1}(x+iy) = A+iB$

or, $\tan(A+iB) = x+iy$

$\therefore \tan(A-iB) = x-iy$

$\therefore \tan 2A = \tan\{(A+iB) + (A-iB)\}$

$= \frac{\tan(A+iB) + \tan(A-iB)}{1 - \tan(A+iB)\tan(A-iB)}$

$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$

or, $2A = \tan^{-1}\dfrac{2x}{1-x^2-y^2}$

$\therefore\ A = \dfrac{1}{2}\tan^{-1}\dfrac{2x}{1-x^2-y^2}$

Also, $\tan(2iB) = \tan\{(A+iB)-(A-iB)\}$

$= \dfrac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)} = \dfrac{x+iy-x+iy}{1+x^2+y^2}$

$= \dfrac{2iy}{1+x^2+y^2}$

$\cancel{\therefore 2iB = \tan^{-1}\dfrac{2iy}{1+x^2+y^2}}$

$\cancel{\text{or, } iB} = \dfrac{1}{2}\tan^{-1}\dfrac{2iy}{1+x^2+y^2}$

or, $i\tanh(2B) = \dfrac{2iy}{1+x^2+y^2}$

or, $\tanh(2B) = \dfrac{2y}{1+x^2+y^2}$

$\therefore\ B = \dfrac{1}{2}\tanh^{-1}\dfrac{2y}{1+x^2+y^2}$

3) Find the real and imaginary part of $(1+i)^{1+i}$.

⇒ We have, $1+i = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

$= \sqrt{2}\, e^{i\frac{\pi}{4}}$ $[\because \cos\theta + i\sin\theta = e^{i\theta}]$

Now, $(1+i)^{1+i} = e^{\log(1+i)^{(1+i)}} = e^{(1+i)\log(1+i)} = e^{(1+i)\log(\sqrt{2}e^{i\frac{\pi}{4}})}$

$= e^{(1+i)\{\frac{1}{2}\log 2 + i\frac{\pi}{4}\}} = e^{(\frac{1}{2}\log 2 - \frac{\pi}{4}) + i(\frac{1}{2}\log 2 + \frac{\pi}{4})}$

$= e^{\frac{1}{2}\log 2 - \frac{\pi}{4}} \cdot e^{i(\frac{1}{2}\log 2 + \frac{\pi}{4})} = e^{\frac{1}{2}\log 2 - \frac{\pi}{4}}\left\{\cos\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\sin\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)\right\}$

$\therefore$ The real part $= e^{\frac{1}{2}\log 2 - \frac{\pi}{4}}\cos\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)$ and imaginary part $= e^{\frac{1}{2}\log 2 - \frac{\pi}{4}}\sin\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)$

Some Results:– i) $x^n - 1 = (x-1)(1+x+x^2+x^3+\dots+x^{n-1})$

or, $1+x+x^2+x^3+\dots+x^{n-1} = \dfrac{x^n-1}{x-1}$

ii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots,\ -1 < x \le 1$

iii) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

iv) $\log\frac{1+x}{1-x} = \log(1+x) - \log(1-x)$

$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$

4) Show that the sum of the 99th power of the roots of the equation $x^5 = 1$ is zero.

$\Rightarrow$ We have, $x^5 = 1 = \cos 0 + i\sin 0 = (\cos 2k\pi + i\sin 2k\pi), k \in \mathbb{Z}$

$\therefore x = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}, \quad k = 0, 1, 2, 3, 4$

$= \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^k, \quad k = 0, 1, 2, 3, 4$

$= \alpha^k, \quad k = 0, 1, \ldots, 4 \quad \left[\text{Let, } \alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right]$

$\therefore$ There are five distinct values of x and they are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

$\therefore 1^{99} + \alpha^{99} + (\alpha^2)^{99} + (\alpha^3)^{99} + (\alpha^4)^{99}$

$= 1 + \alpha^{99} + (\alpha^{99})^2 + (\alpha^{99})^3 + (\alpha^{99})^4$

$= \frac{(\alpha^{99})^5 - 1}{\alpha^{99} - 1} = \frac{(\alpha^5)^{99} - 1}{\alpha^{99} - 1} = \frac{1-1}{\alpha^{99} - 1} = 0$ (Proved)

$[\because \alpha^5 = 1 \text{ and } \alpha^{99} \neq 1]$

5) Gregory's series:-

We have, $i\tan x = \frac{i\sin x}{\cos x}$

or, $\frac{i\sin x}{\cos x} = \frac{i\tan x}{1}$

or, $\frac{\cos x + i\sin x}{\cos x - i\sin x} = \frac{1 + i\tan x}{1 - i\tan x}$

or, $\frac{e^{ix}}{e^{-ix}} = \frac{1 + i\tan x}{1 - i\tan x} = e^{2ix}$

or, $2ix = \log\frac{1+i\tan x}{1-i\tan x} = 2\left\{ i\tan x - i\frac{\tan^3 x}{3} + i\frac{\tan^5 x}{5} - i\frac{\tan^7 x}{7} + \cdots \right.$

or, $\boxed{x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \frac{\tan^7 x}{7} + \cdots}$, where $|\tan x| < 1$

also, $\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}$ [Putting $x = \tan^{-1} x$]

6) Show that $\frac{\pi}{8} = \frac{1}{1\cdot3} + \frac{1}{5\cdot7} + \frac{1}{9\cdot11} + \cdots$

⇒ We have the Gregory's series,

$$x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} - \frac{\tan^{11} x}{11} + \cdots$$

Putting $x = \frac{\pi}{4}$ in gregory's series we get,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

$$= \frac{2}{3\cdot1} + \frac{2}{5\cdot7} + \frac{2}{9\cdot11} + \cdots$$

or, $\frac{\pi}{8} = \frac{1}{3\cdot1} + \frac{1}{5\cdot7} + \frac{1}{9\cdot11} + \cdots$ (Proved)

7) Show that $\frac{\pi}{4} = \left[\frac{2}{3} + \frac{1}{7}\right] - \frac{1}{3}\left[\frac{2}{3^3} + \frac{1}{7^3}\right] + \frac{1}{5}\left[\frac{2}{3^5} + \frac{1}{7^5}\right] + \cdots$

⇒ We have the Gregory's series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \text{ ---- (i)}$$

$$\therefore \tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3\cdot3^3} + \frac{1}{5\cdot3^5} - \frac{1}{7\cdot3^7} + \cdots$$

$$\therefore \tan^{-1}\frac{1}{7} = \frac{1}{7} - \frac{1}{3\cdot7^3} + \frac{1}{5\cdot7^5} - \frac{1}{7\cdot7^7} + \cdots$$

Now, given expression,

$$\left[\frac{2}{3} + \frac{1}{7}\right] - \frac{1}{3}\left[\frac{2}{3^3} + \frac{1}{7^3}\right] + \frac{1}{5}\left[\frac{2}{3^5} + \frac{1}{7^5}\right] + \cdots$$

$$= 2\left[\frac{1}{3} - \frac{1}{3\cdot3^3} + \frac{1}{5\cdot3^5} - \cdots\right] + \left[\frac{1}{7} - \frac{1}{3\cdot7^3} + \frac{1}{5\cdot7^5}\right.$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{2/3}{1-1/9} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3/4 + 1/7}{1 - 3/4 \times 1/7} = \tan^{-1}\frac{25}{25} = \tan^{-1}(1) = \frac{\pi}{4} \text{ (Proved)}$$

8) Find the smallest positive integer n such that $\left(\frac{1+i}{1-i}\right)^n = 1$

$\Rightarrow$ $\frac{1+i}{1-i} = \frac{(1+i)^2}{1+1} = \frac{1+2i-1}{2} = i$

$\therefore$ Now, $\left(\frac{1+i}{1-i}\right)^n = 1$

or, $i^n = 1$

$\therefore$ Smallest value of n is 4

9) Find the general and Principle value of $(1-i)^{1+i}$

$\Rightarrow$ We have,

$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$= \cancel{\sqrt{2}\cos}\ \sqrt{2}\, e^{-i\frac{\pi}{4}}$

$\therefore (1-i)^{1+i} = e^{\operatorname{Log}(1-i)^{1+i}} = e^{(1+i)\operatorname{Log}(1-i)}$

$= e^{(1+i)\{\log(1-i) + 2n\pi i\}}$

$= e^{(1+i)\{2n\pi i + \log\sqrt{2}\, e^{-i\frac{\pi}{4}}\}}$

$= e^{(1+i)\{2n\pi i + \frac{1}{2}\log 2 - i\frac{\pi}{4}\}}$

$= e^{(1+i)\{\frac{1}{2}\log 2 + (2n - \frac{1}{4})\pi i\}}$

$= e^{\{\frac{1}{2}\log 2 - (8n-1)\frac{\pi}{4}\} + i\{\frac{1}{2}\log 2 + (8n-1)\frac{\pi}{4}\}}$

$= e^{\frac{1}{2}\log 2 - (8n-1)\frac{\pi}{4}} \cdot e^{i\{\frac{1}{2}\log 2 + (8n-1)\frac{\pi}{4}\}}$

$= e^{\frac{1}{2}\log 2 - (8n-1)\frac{\pi}{4}} \left[\cos\left\{\frac{1}{2}\log 2 + (8n-1)\frac{\pi}{4}\right\} + i\sin\left\{\frac{1}{2}\log 2 + (8n-1)\frac{\pi}{4}\right\}\right]$

$\therefore$ The general value is $e^{\frac{1}{2}\log 2 - (8n-1)\frac{\pi}{4}} \left[\cos\left\{\frac{1}{2}\log 2 + (8n-1)\frac{\pi}{4}\right\} + i\sin\left\{\frac{1}{2}\log 2 + (8n-1)\frac{\pi}{4}\right\}\right]$

$\therefore$ Principle value $= e^{\frac{1}{2}\log 2 + \frac{\pi}{4}} \left\{\cos\left(\frac{1}{2}\log 2 - \frac{\pi}{4}\right) + i\sin\left(\frac{1}{2}\log 2 - \frac{\pi}{4}\right)\right\}$

12)

$\Rightarrow$

10) Show that all the values of $(i)^i$ are real.

$\Rightarrow$ Now, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$\therefore (i)^i = e^{\log(i)^i} = e^{i\log i} = e^{i(2n\pi i + \log i)}$

$= e^{i(2n\pi i + \log e^{i\pi/2})} = e^{i(2n\pi i + i\pi/2)} = e^{-(2n\pi + \pi/2)} = e^{-(4n+1)\pi/2}$, $n \in \mathbb{Z}$

11) If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$ and $\cos\phi = \frac{1}{2}\left(b + \frac{1}{b}\right)$ then show that $\cos(\theta+\phi)$ is one of the values of $\frac{1}{2}\left(ab + \frac{1}{ab}\right)$.

$\Rightarrow$ $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$

or, $a^2 - 2a\cos\theta + 1 = 0$

$\therefore a = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$

Simmilarly from $\cos\phi = \frac{1}{2}\left(b + \frac{1}{b}\right)$ we get,

$b = \cos\phi \pm i\sin\phi$.

Taking positive sign, $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$

$\therefore \frac{1}{a} = \cos\theta - i\sin\theta$, $\frac{1}{b} = \cos\phi - i\sin\phi$

$\therefore \frac{1}{ab} = (\cos\theta - i\sin\theta)(\cos\phi - i\sin\phi) = \cos(\theta+\phi) - i\sin(\theta+\phi)$

$\therefore ab = \cos(\theta+\phi) + i\sin(\theta+\phi)$

$\therefore ab + \frac{1}{ab} = 2\cos(\theta+\phi)$

or, $\cos(\theta+\phi) = \frac{1}{2}\left(ab + \frac{1}{ab}\right)$

12) Show that the solution of the equation $\cos x = 2$ is given by $x = 2n\pi \pm i\log(2+\sqrt{3})$.

$\Rightarrow$ $\cos x = 2$

or $\dfrac{e^{ix} + e^{-ix}}{2} = 2$

or, $e^{ix} + e^{-ix} = 4$

Let, $e^{ix} = a$

$\therefore a + \frac{1}{a} = 4$

or, $a^2 - 4a + 1 = 0$

or, $a = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

taking positive sign,

$\cos a = 2+\sqrt{3}$

or, $e^{ix} = 2+\sqrt{3}$

or, $ix = \operatorname{Log}(2+\sqrt{3}) = 2n\pi i + \log(2+\sqrt{3})$

or, $x = 2n\pi - i\log(2+\sqrt{3})$

taking negative sign,

$a = 2-\sqrt{3}$

$e^{ix} = 2-\sqrt{3}$

or $ix = \operatorname{Log}(2-\sqrt{3}) = 2n\pi i + \log(2-\sqrt{3})$

or, $x = 2n\pi - i\log(2-\sqrt{3})$

$= 2n\pi + i\log(2-\sqrt{3})^{-1}$

$= 2n\pi + i\log\frac{1}{2-\sqrt{3}}$

$= 2n\pi + i\log(2+\sqrt{3})$

$\therefore x = 2n\pi \pm i\log(2+\sqrt{3})$

13) Show that the roots of the equations $\sin z = 0$ are all real, where z is a complex number.

$\Rightarrow$ Let, $z = x+iy$.

$\therefore \sin z = 0$

or, $\sin(x+iy) = 0$

or, $\sin x\cos(iy) + \cos x\sin(iy) = 0$

or, $\sin x\cosh y + i\sinh y\cos x = 0$

$\therefore \sin x\cosh y = 0$ — (i) and $\sinh y\cos x = 0$ — (ii)

from (i), $\sin x = 0$ ($\because \cosh y \neq 0$ for any real value of y)

$\therefore x = n\pi$

$\therefore$ from (ii), $\sinh y = 0$

$\therefore y = 0$

$\therefore$ All the roots of the equation $\sin z = 0$ are real.

14) Show that $\sin\left[i \log \frac{a-ib}{a+ib}\right] = \frac{2ab}{a^2+b^2}$.

=> Let, $a = r\cos\theta$, $b = r\sin\theta$

$\therefore r = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$

$\therefore (a+ib) = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore (a-ib) = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

$\therefore \sin\left[i\log\frac{a-ib}{a+ib}\right] = \sin\left[i\log e^{-2i\theta}\right] = \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$= \frac{2\, b/a}{1+b^2/a^2} = \frac{2ab}{a^2+b^2}$

Solve the equation $e^{4z} = i$, where z is a complex number.

Let, $z = x+iy$. (x and y both are real)

$\therefore e^{4z} = i$

or, $e^{4x+4iy} = i$

or, $e^{4x} \cdot e^{4iy} = i$

or, $e^{4x}(\cos 4y + i\sin 4y) = i$

$\therefore e^{4x}\cos 4y = 0$ —— (i) and $e^{4x}\sin 4y = 1$ —— (ii)

squaring and adding (i) and (ii) we have,

$e^{8x} = 1$

or, $x = 0$

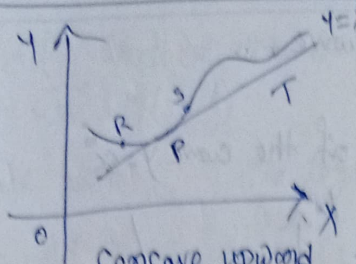
$\therefore$ from (i),

$\cos 4y = 0$

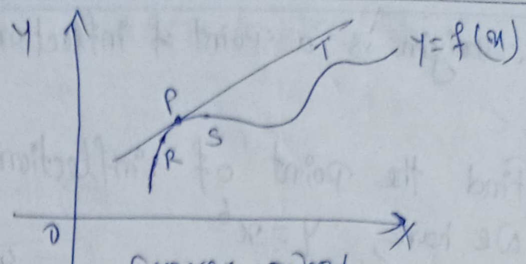
$\therefore 4y = \frac{\pi}{2}$

or, $y = \frac{\pi}{8}$

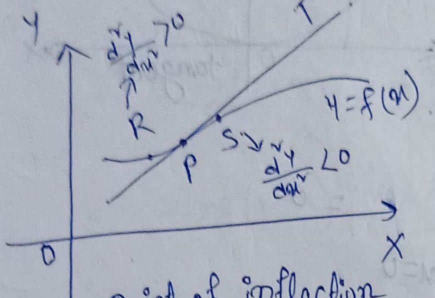
Point of Inflection (Inflexion)



Concave upward or Convex downward $\left.\frac{d^2y}{dx^2}\right|_{N(P)} > 0$

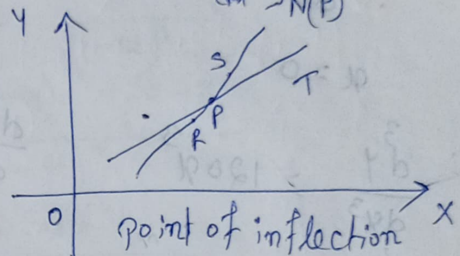


Convex upward or Concave downward



Point of inflection

$\left.\frac{d^2y}{dx^2}\right|_{N(P)} < 0$



Point of inflection

[NOTE :- 1) If $\frac{d^2y}{dx^2}$ changes it's sign ~~from~~ as x passes the point P from left to right. then the point P is called point of inflection.

2) The tangent at the point of inflection cuts the curve instead of ~~touchig~~ touching it.

3) If for the curve $y=f(x)$, $\left.\frac{d^2y}{dx^2}\right|_{x=a} = \left.\frac{d^3y}{dx^3}\right|_{x=a} = \cdots = \left.\frac{d^{n-1}y}{dx^{n-1}}\right|_{x=a} = 0$ but $\left.\frac{d^ny}{dx^n}\right|_{x=a} \neq 0$. If 'n' is odd then the point $x=a$ is a Point of inflection.]

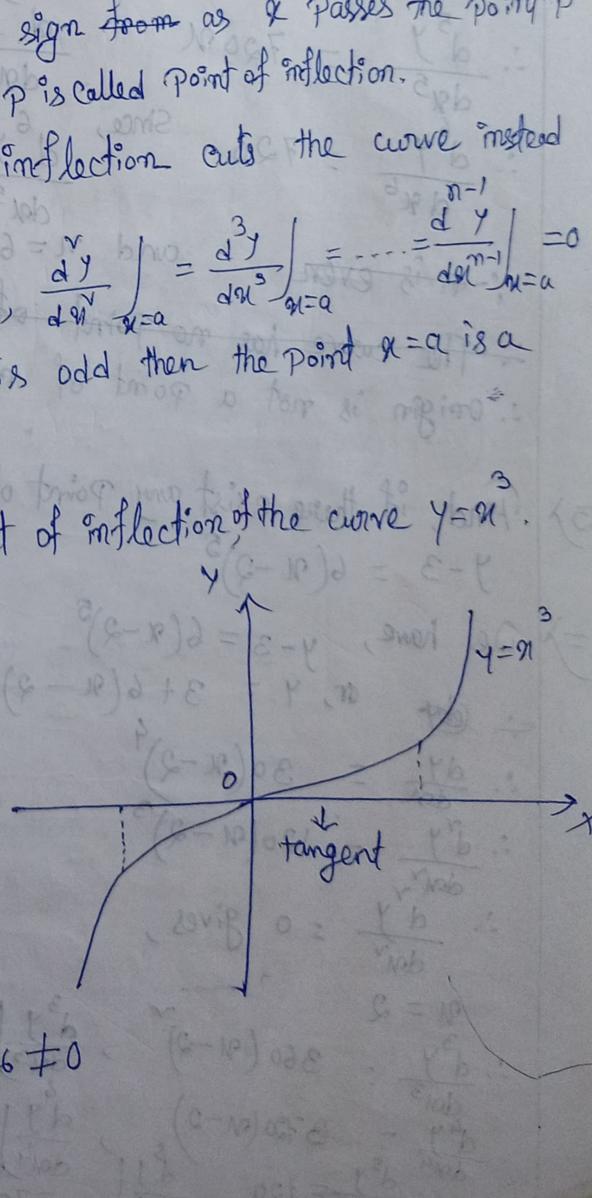
1) Show that origin is the point of inflection of the curve $y=x^3$.

$\Rightarrow$ We have, $y=x^3$.

$\therefore \frac{dy}{dx} = 3x^2$

$\frac{d^2y}{dx^2} = 6x$

$\frac{d^3y}{dx^3} = 6$



Now, $\frac{d^2y}{dx^2} = 0$ gives,

$6x = 0$

or, $x = 0$.

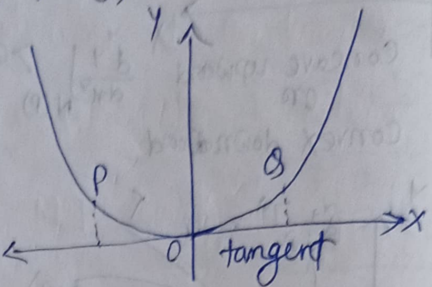
$\therefore \left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$, $\left.\frac{d^3y}{dx^3}\right|_{x=0} = 6 \neq 0$

∴ Origin is a Point of inflection of the curve.

2) Find the point of inflection, if any, of the curve $y = x^6$.

⇒ We have, $y = x^6$

∴ $\frac{dy}{dx} = 6x^5$, $\frac{d^2y}{dx^2} = 30x^4$



now, $\frac{d^2y}{dx^2} = 0$ gives,

$x = 0$

∴ $\frac{d^3y}{dx^3} = 120x^3$ $\qquad \left.\frac{d^3y}{dx^3}\right|_{x=0} = 0$

∴ $\frac{d^4y}{dx^4} = 360x^2$ $\qquad \left.\frac{d^4y}{dx^4}\right|_{x=0} = 0$

∴ $\frac{d^5y}{dx^5} = 720x$ $\qquad \left.\frac{d^5y}{dx^5}\right|_{x=0} = 0$

∴ $\frac{d^6y}{dx^6} = 720$ $\qquad$ Since, $\left.\frac{d^6y}{dx^5}\right|_{x=0} = 720 \neq 0$

~~∴ n is even~~ $\qquad$ and $n = 6$ is even.

~~∴ The curve has no point of reflection.~~

∴ Origin is not a point of reflection of the curve.

3) Find, if there exist any point of inflection on the curve $y - 3 = 6(x-2)^5$

⇒ We have, $y - 3 = 6(x-2)^5$

or, $y = 3 + 6(x-2)^5$

∴ $\frac{dy}{dx} = 30(x-2)^4$ $\qquad$ ∵ $n = 5$ is odd

∴ $\frac{d^2y}{dx^2} = 120(x-2)^3$ $\qquad$ ∴ $(2,3)$ is a point of inflection.

∴ $\frac{d^2y}{dx^2} = 0$ gives,

$x = 2$

∴ $\frac{d^3y}{dx^3} = 360(x-2)^2$, $\left.\frac{d^3y}{dx^3}\right|_{x=2} = 0$

$\frac{d^4y}{dx^4} = 720(x-2)$, $\left.\frac{d^4y}{dx^4}\right|_{x=2} = 0$

$\frac{d^5y}{dx^5} = 720$, $\left.\frac{d^5y}{dx^5}\right|_{x=2} = 720 \neq 0$

4) Verify whether the origin is the point of inflection of the curve $y = x^2 \log(1-x)$

$\Rightarrow$ We have, $y = x^2 \log(1-x)$

$$\therefore \frac{dy}{dx} = 2x\log(1-x) - \frac{x^2}{1-x}$$

$$\therefore \frac{d^2y}{dx^2} = 2\log(1-x) - \frac{2x}{1-x} - \frac{2x(1-x)+x^2}{(1-x)^2}$$

$$= 2\log(1-x) - \frac{2x}{1-x} - \frac{2x-x^2}{(1-x)^2}$$

$$= 2\log(1-x) - \frac{2x(1-x)+2x-x^2}{(1-x)^2}$$

$$= 2\log(1-x) - \frac{4x-3x^2}{(1-x)^2}$$

Now, $\frac{d^2y}{dx^2} = 0$ gives,

$$2\log(1-x) - \frac{4x-3x^2}{(1-x)^2} = 0$$

or, $2(1-x)^2 \log(1-x) - 4x + 3x^2 = 0$

or, $2(1+x^2-2x)\log(1-x) - 4x + 3x^2 = 0$

or, $(2+2x^2-4x)\log(1-x) - 4x + 3x^2 = 0$

or, $x = 0$

$$\frac{d^3y}{dx^3} = -\frac{2}{1-x} - \frac{(1-x)^2(4-6x) + 2(4x-3x^2)(1-x)}{(1-x)^4}$$

$$\frac{d^3y}{dx^3} = -2 - 4 = -6 \neq 0$$

$\therefore n = 3$ is odd.

$\therefore (0,0)$ is a point of inflection.

5) Find the points of inflections, if any, of the curve $x=(\log y)^3$

$\Rightarrow$ We have, $x=(\log y)^3$

$\therefore \frac{dx}{dy} = \frac{3(\log y)^2}{y}$

$\therefore \frac{d^2x}{dy^2} = \cancel{\frac{6\log y}{y} - \frac{3(\log y)^2}{y^2}}$

$= \frac{6\log y \cdot \frac{1}{y} \cdot y - 3(\log y)^2 \cdot 1}{y^2}$

$= \frac{6\log y - 3(\log y)^2}{y^2}$

$\therefore \frac{d^2x}{dy^2} = 0$ gives,

$6\log y - 3(\log y)^2 = 0$

or, $3\log y\,(2-\log y) = 0$

$\therefore \log y = 0$ | $\log y = 2$

$y=1$ | $y = e^2$

$\therefore x = 0$ | $\therefore x = 8$

$\therefore \frac{d^3x}{dy^3} = \cancel{\frac{y\frac{6}{y} - 3(\log y)^2\, 2y}{y^4}} = \cancel{6(1-\log y)}$

$= \frac{y^2\left(\frac{6}{y} - \frac{6\log y}{y}\right) - 2y\{6\log y - 3(\log y)^2\}}{y^4}$

$= \frac{6(1-\log y) - 6\{3\log y - (\log y)^2\}}{y^3}$

$= \frac{6\{1-\log y - 3\log y + (\log y)^2\}}{y^3}$

$= \frac{6\{1 - 4\log y + (\log y)^2\}}{y^3}$ $\quad 1-8+4 \quad 1-4 \quad -3 \quad -18$

$\left.\frac{d^3x}{dy^3}\right]_{y=1} = 6 \neq 0$, $\left.\frac{d^3x}{dy^3}\right]_{y=e^2} \neq 0$

$\therefore y=1$ and $y=e^2$ gives the point of inflection.

We have,
when $y=1$, $x=0$ and $y=e^2$, $x=8$.
$\therefore (0,1)$ and $(8, e^2)$ are the points of inflection.

6) Investigate the curve $y=(x-1)^{1/3}$ for point of inflection.

$\Rightarrow$ We have, $y=(x-1)^{1/3}$

$\therefore \frac{dy}{dx} = \frac{1}{3}(x-1)^{-2/3}, x \neq 1$, $\frac{d^2y}{dx^2} = -\frac{2}{9}(x-1)^{-5/3}, x \neq 1$

$\frac{d^2y}{dx^2}=0$ gives,

$(x-1)^{-5/3} = 0$

~~or, $x=1$~~

or, $\frac{1}{(x-1)^{5/3}}=0$

$\frac{d^2y}{dx^2} \neq 0$ for any value of x.

$\frac{d^2y}{dx^2} = -\frac{2}{9(x-1)^{5/3}}$

we have,
$\frac{d^2y}{dx^2} > 0$, when $x<1$

and $\frac{d^2y}{dx^2} < 0$, when $x>1$

~~We have $\frac{d^2y}{dx^2}$~~

Since, $\frac{d^2y}{dx^2}$ changes it's sign as x passes 1 from left to right.

$\therefore x=1$ is a point of inflection.

7) Find, if there exist any point on inflection on the curve $y-5=8(x-7)^6$ [$x=7$ Ans:-]

8) Find the point of inflection, if any, of the curve $y=\frac{x^3}{a^2+x^2}$ [$x=0$, $x=\sqrt{3}a$, $x=-\sqrt{3}a$ Ans!]

9) Find the point of inflection of the curve, $y^2=(x-a)^2(x-b)$ and show that the the point of inflection lies on the line $3x+a=4b$ [$x=\frac{4b-a}{3}$]

10) ~~Find the point~~ Show that, the curve $y=\frac{1-x}{1+x^2}$ has 3 points of inflections which lies on a straight line. [$x=-1, 2+\sqrt{3}, 2-\sqrt{3}$]

11) Show that, the line joining the two points of inflection of the curve $y^2(x-a)=x^2(x+a)$ subtained an angle $\frac{\pi}{3}$ at the origin.

7) The given curve,
$$y-5 = 8(x-7)^6$$
or, $y = 5+8(x-7)^6$

$\therefore \frac{dy}{dx} = 48(x-7)^5$, $\frac{d^2y}{dx^2} = 48\times5(x-7)^4$

$\frac{d^2y}{dx^2} = 0$ gives,

$48\times5(x-7)^4 = 0$

or, $x = 7$.

$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=7} = 0$

$\frac{d^3y}{dx^3} = 48\times5\times4(x-7)^3$ $\left.\frac{d^3y}{dx^3}\right]_{x=7} = 0$

$\frac{d^4y}{dx^4} = 48\times5\times4\times3(x-7)^2$ $\left.\frac{d^4y}{dx^4}\right]_{x=7} = 0$

$\frac{d^5y}{dx^5} = 48\times5\times4\times3\times2(x-7)$ $\left.\frac{d^5y}{dx^5}\right]_{x=7} = 0$

$\frac{d^6y}{dx^5} = 48\times5\times4\times3\times2$ $\left.\frac{d^6y}{dx^6}\right]_{x=7} \neq 0$

But, $n=6$ is even

$\therefore (7,5)$ is not a point of reflection.

8) given curve, $y = \frac{x^3}{a^2+x^2}$

$\therefore \frac{dy}{dx} = \frac{3x^2(a^2+x^2)-2x^3 x}{(a^2+x^2)^2} = \frac{3a^2x^2+3x^4-2x^4}{(a^2+x^2)^2} = \frac{3a^2x^2+x^4}{(a^2+x^2)^2}$

$\therefore \frac{d^2y}{dx^2} = \frac{(6a^2x+4x^3)(a^2+x^2)^2-2(3a^2x^2+x^4)(a^2+x^2)2x}{(a^2+x^2)^4}$

$= \frac{(6a^2x+4x^3)(a^4+x^4+2a^2x^2)-4x(3a^4x^2+3a^2x^4+a^2x^4+x^6)}{(a^2+x^2)^4}$

$= \frac{6a^6x+6a^2x^5+12a^4x^3+4x^3a^4+\cancel{4x^7}+8a^2x^5-12x^3a^4-12x^5a^2-4a^2x^5-\cancel{4x^7}}{(a^2+x^2)^4}$

$= \frac{6a^2x-2a^2x^5+4x^3a^4}{(a^2+x^2)^4}$

$\frac{d^2y}{dx^2} = 0$ gives,

$6a^2x - 2a^2x^5+4x^3a^4 = 0$

or, $6x-2x^5+4x^3a^2 = 0$

or, $x(3-x^4+2x^2a^2) = 0$

$\therefore x = 0$

$x^4 - 2a^2x^2 - 3 = 0$

$x^2 = \frac{2a^2 \pm \sqrt{4a^4+12}}{2}$

$= a^2 \pm \sqrt{a^4+3}$

9) The given curve,

$$y^2 = (x-a)^2 (x-b)$$

or, $y = (x-a)\sqrt{x-b}$

$$\therefore \frac{dy}{dx} = \sqrt{x-b} + \frac{x-a}{2\sqrt{x-b}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{x-b}} + \frac{1}{2}\,\frac{\sqrt{x-b} - (x-a)\frac{1}{2\sqrt{x-b}}}{x-b}$$

$$= \frac{1}{2\sqrt{x-b}} + \frac{2x-2b-x+a}{2(x-b)^{3/2}} = \frac{1}{2\sqrt{x-b}} + \frac{x+a-2b}{4(x-b)^{3/2}}$$

$$= \frac{(x-b)^{-1/2}}{2} + \frac{x-a-2b}{4(x-b)^{3/2}}$$

$\frac{d^2y}{dx^2} = 0$ gives,

$2(x-b) + x+a-2b = 0$

or, $3x + a - 4b = 0$

or, $x = \frac{4b-a}{3}$

$$\frac{d^3y}{dx^3} = -\frac{1}{4(x-b)^{3/2}} + \frac{(x-b)^{3/2} - \frac{3}{2}(x+a-2b)(x-b)^{1/2}}{4(x-b)^3}$$

$$\left.\frac{d^3y}{dx^3}\right]_{x=\frac{4b-a}{3}} = \cancel{-\frac{1}{4\left(\frac{4b-a}{3}-b\right)^{3/2}}} + \cancel{\frac{(x-b)^{3/2} - \frac{3}{2}(x-b)(x-a-2b)^{1/2}}{4(x-b)^3}}$$

$$= -\frac{1}{4\left(\frac{4b-a}{3}-b\right)^{3/2}} + \frac{\left(\frac{4b-a}{3}-b\right)^{3/2} - \frac{3}{2}\left(\frac{4b-a}{3}+a-2b\right)\sqrt{\frac{4b-a}{3}-b}}{4\left(\frac{4b-a}{3}-b\right)^3}$$

$$= -\frac{1}{4\left(\frac{4b-a-3b}{3}\right)^{3/2}} + \frac{\left(\frac{4b-a-3b}{3}\right)^{3/2} - \frac{3}{2}\left(\frac{4b-a+3a-6b}{3}\right)\sqrt{\frac{4b-a-3b}{3}}}{4\left(\frac{4b-a-3b}{3}\right)^3}$$

$$= \frac{-1}{4\left(\frac{b-a}{3}\right)^{3/2}} + \frac{\left(\frac{b-a}{3}\right)^{3/2} - \frac{2a-2b}{2}\sqrt{\frac{b-a}{3}}}{4\left(\frac{b-a}{3}\right)^3} \neq 0$$

$n=3$ is odd

$$\therefore y = \left(\frac{4b-a}{3}-a\right)\sqrt{\frac{4b-a}{3}-b} = \frac{4b-4a}{3}\sqrt{\frac{b-a}{3}} = \frac{4(b-a)^{3/2}}{3\sqrt{3}}$$

$\therefore \left(\frac{4b-a}{3}, \frac{4(b-a)^{3/2}}{3\sqrt{3}} \right)$ is the point of inflection.

10) The given curve, $y = \frac{1-x}{1+x^2}$

$$\therefore \frac{dy}{dx} = \frac{-1(1+x^2) - 2x(1-x)}{(1+x^2)^2} = \frac{-1-x^2-2x+2x^2}{(1+x^2)^2}$$

$$= \frac{x^2-2x-1}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(2x-2)(1+x^2)^2 - 2(x^2-2x-1)(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{(2x-2)(1+x^4+2x^2) - 4x(\cancel{x^2}-2x-1+x^4-2x^3-\cancel{x^2})}{(1+x^2)^4}$$

$$= \frac{2x-2+2x^5-2x^4+4x^3-4x^2-\cancel{4x^3}+8x^2+4x-4x^5+8x^4+4x^3}{(1+x^2)^4}$$

$$= \cancel{\frac{-2x^5+6x^4+4x^3+8x^2+2x-2}{(1+x^2)^4}} = \frac{-2x^5+6x^4+4x^3+4x^2+6x-2}{(1+x^2)^4}$$

$\frac{d^2y}{dx^2} = 0$ gives,

$x^5 - 3x^4 - 2x^3 - 4x^2 \cancel{- 3x + 2 = 0}$

11)

11) given equation,

$$y = \frac{x\sqrt{x+a}}{\sqrt{x-a}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x-a}\left[\sqrt{x+a} + \frac{x}{2\sqrt{x+a}}\right] - \frac{x\sqrt{x+a}}{2\sqrt{x-a}}}{(x-a)}$$

$$= \frac{\sqrt{x^2-a^2} + \frac{x\sqrt{x-a}}{2\sqrt{x+a}} - \frac{x\sqrt{x+a}}{2\sqrt{x-a}}}{x-a}$$

$$= \frac{2x^2-2a^2 + x(x-a) - x(x+a)}{2(x-a)\sqrt{x^2-a^2}}$$

$$= \frac{2x^2-2a^2 + \cancel{x^2} - xa - \cancel{x^2} - ax}{2(x-a)\sqrt{x^2-a^2}} = \frac{x^2-a^2-ax}{(x-a)\sqrt{x^2-a^2}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(2x-a)(x-a)\sqrt{x^2-a^2} - (x^2-a^2-ax)\left\{\sqrt{x^2-a^2} + \frac{(x-a)x}{\sqrt{x^2-a^2}}\right\}}{(x-a)^2(x^2-a^2)}$$

$$= \frac{(2x-a)(x-a)\sqrt{x^2-a^2} - (x^2-a^2-ax)\frac{x^2-a^2+x^2-ax}{\sqrt{x^2-a^2}}}{(x-a)^2(x^2-a^2)}$$

$$= \frac{(2x-a)(x-a)(x^2-a^2) - (x^2-a^2-ax)(2x^2-a^2-ax)}{(x-a)^2(x^2-a^2)\sqrt{x^2-a^2}}$$

$\frac{d^2y}{dx^2} = 0$ gives,

$x = -2a$

$\left.\frac{d^3y}{dx^3}\right]_{x=-2a} \neq 0$ and $x = 3$ is odd

When $x = -2a$

$$y = \frac{-2a\sqrt{-a}}{\sqrt{-3a}} = \pm\frac{2a}{\sqrt{3}}$$

$\therefore \left(-2a, \pm\frac{2a}{\sqrt{3}}\right)$ are the points of inflection.

$\therefore$ ~~$\left(-2a, \frac{2a}{\sqrt{3}}\right)$~~ The lines are $= \sqrt{4a^2 + \frac{4a^2}{3}}$

$$= \sqrt{\frac{16a^2}{3}} = \frac{4a}{3}$$